stratosph.com
contact@stratosph.com

Decentralized Finance (DeFi)

Part II

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stratosph.com
contact@stratosph.com

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Decentralized exchanges

Uniswap: https://uniswap.org/

We will discuss Uniswap as it's currently the leader in the DEX space. We will present its major innovations and some basic mathematics behind it.

Uniswap is built on Ethereum. Due to smart contracts limitations, an order book model appears to be inferior from an user experience point of view, at least when directly implemented on layer 1. The speed performance (dictated by the blocktime) and segregation between order posting and matching vs settlement is a barrier for standard users (although currently scaling solutions are starting on layer 2 with other technologies such as ZK-Rollup, Optimism, StarkWare etc; they are offering a new class of DEXes with better speed and ease of use, but none has become as popular as Uniswap).

Uniswap has come up with a solution on how to organize markets without order books. It put in place the concept of Automated market makers (AMM) and Constant product market (CPM), which we discuss below.

Automated market makers

In order book models, the market makers are active: they post orders and manage them individually. In the AMM model, the market makers are pooled together and are passive. Their assets are put into pools which are available at any moment for takers to trade. The market makers are treated the same (since their assets are pooled together) and they don't have to actively manage their assets (hence the name "automated"). For example, for each trading pair A/B, there would be a pool for asset A and a pool for asset B. The price adjusts depending on the amount of the remaining assets in each pool.

The pricing formula needs to satisfy some mathematical properties so that one gets a functioning market. The CPM, a surprisingly simple yet powerful model, offers what is needed to make this possible.

Constant product market

Liquidity providers (LPs) supply assets α and β to 2 pools with respective quantities R_α and R_β.

A constant k is fixed and the product of the asset quantities must always equal to k:

$$k = R_\beta^t R_\alpha^t, \forall t$$

A trader (or an arbitrageur) decides to sell Δ_α of the coin α in exchange for Δ_β of the coin β. After her trade the pool α will have $R_\alpha + \Delta_\alpha$ and the pool β will have $R_\beta - \Delta_\beta$.

The constant product is an invariant and now writes (we drop the superscript):

$$k = (R_\alpha + \Delta_\alpha)(R_\beta - \Delta_\beta)$$

The quantity of coin β she will receive is thus:

$$\Delta_\beta = R_\beta - \frac{k}{R_\alpha + \Delta_\alpha} = \frac{R_\beta \Delta_\alpha}{R_\alpha + \Delta_\alpha}$$

Because of her trade amount, her realised price is:

$$\hat{m} = \frac{\Delta_\beta}{\Delta_\alpha} = \frac{R_\beta}{R_\alpha + \Delta_\alpha}$$

When the trade amount is infinitesimal the price is simply the ratio of the reserve quantities:

$$\lim_{\Delta_\alpha \to 0} \hat{m} = \frac{R_\beta}{R_\alpha} = m$$

If the trader has the knowledge of an external price p, she sees an arbitrage opportunity and is incentivized to close the arbitrage by trading with Uniswap. Her PnL after the arbitrage is:

$$g = \Delta_\beta - \Delta_\alpha p = \frac{R_\beta \Delta_\alpha}{R_\alpha + \Delta_\alpha} - \Delta_\alpha p$$

She executes an optimal arbitrage when her PnL is maximized:

$$\Delta_\alpha^* = \text{argmax } g(\Delta_\alpha) = \text{argmax}\left(\frac{R_\beta \Delta_\alpha}{R_\alpha + \Delta_\alpha} - \Delta_\alpha p\right)$$

This is reached when Δ_α^*. cancels the derivative of g, i.e. Δ_α^* is such that:

$$g'(\Delta_\alpha^*) = \frac{k}{(R_\alpha + \Delta_\alpha^*)^2} - p = 0$$
$$\Delta_\alpha^* = \sqrt{k/p} - R_\alpha$$

Δ_α^* is her optimal trade quantity where she maximizes her profit. The value of the pool after the optimal arbitrage is:

$$V_\alpha = (R_\alpha + \Delta_\alpha^*)p = \sqrt{kp}, \qquad V_\beta = R_\beta - \Delta_\beta^* = \frac{k}{R_\alpha + \Delta_\alpha^*} = \sqrt{kp}$$

So the optimal arbitrage is such that the value of each pool is restored to have the same value under the new price p.

When the arbitrage is removed (i.e. there is no longer an arbitrage opportunity), the price given by Uniswap is equal to the external price:

$$m = \frac{R_\beta - \Delta_\beta^*}{R_\alpha + \Delta_\alpha^*} = \frac{k}{(R_\alpha + \Delta_\alpha^*)^2} = p$$

"Impermanent loss"

As noted above, under no-arbitrage assumption, at time t the price m of α in unit of β is given by the ratio of the pooled quantities:

$$m^t = R_\beta^t / R_\alpha^t$$

Note that to correctly reflect prices, R_β and R_α change over time as the result of trading or arbitraging, but must respect the Constant Product k:

$$k = R_\beta^t R_\alpha^t, \forall t$$

In particular take $t = 0$ gives:

$$k = (R_\alpha^0)^2 m^0$$

The total value of the pools at time t is the sum of each pool:

$$V^t = R_\beta^t + m^t R_\alpha^t = 2R_\beta^t = 2R_\alpha^t m^t$$

The last equalities say that each pool has the same value. This comes from the CPM construction and the no-arbitrage assumption.

Note that the no-arbitrage assumption generally verifies in practice because arbitrageurs are incentivized to restore the correct price should the Uniswap price deviate from the latter (in fact the correct price is where the arbitrage profit is maximized, see above).

At a time horizon T the total value of the pools is:

$$V^T = 2R_\beta^T = 2\sqrt{(R_\beta^T R_\alpha^T)(R_\beta^T / R_\alpha^T)} = 2\sqrt{km^T} = 2\sqrt{(R_\alpha^0)^2 m^0 m^T} = 2R_\alpha^0 \sqrt{m^0 m^T}$$

This form is convenient because it expresses the value of the pools in terms of the terminal price m^T.
If the LPs doesn't provide assets to the pools but keep them to T, she will have (we assume she still split the asset into α and β):

$$\overline{V}^T = R_\alpha^0 m^T + R_\beta^0 = R_\alpha^0 (m^0 + m^T)$$

The Impermanent Loss, i.e. the difference in value between providing assets to the pools versus doing nothing, is given by:

$$L^T = V^T - \overline{V}^T = R_\alpha^0 (2\sqrt{m^0 m^T} - (m^0 + m^T))$$

If we denote $r = m^T / m^0$ we get:

$$L^T = R_\alpha^0 m^0 (2\sqrt{r} - (r+1)) = (2R_\alpha^0 m^0)\left(\frac{2\sqrt{r} - (r+1)}{2}\right)$$

The percentage loss to the initial capital is $\frac{2\sqrt{r}-(r+1)}{2}$.

$V^T(m^T)$ as a function of the terminal price m^T can be replicated statically by a portfolio of linear derivatives and a basket of vanilla options. If we assume the interest rate is 0 and the martingality of the asset price (i.e. $E(m^T) = m^0$), the well known Carr formula permits to write (we drop the superscript):

$$V(m^T) = V(m^0) + V'(m^0)(m^T - m^0) + \int_{m^0}^{\infty} V''(u) C(u) du + \int_0^{m^0} V''(u) P(u) du$$

where $C(u)$ is the payoff of a European Call option of maturity T and strike u (resp. $P(u)$ denotes the European Put payoff).
The derivatives of $V(.)$ are straightforward to compute:

$$V(m^0) = 2\sqrt{km^0}, \qquad V'(m^0) = \sqrt{k/m^0}, \qquad V''(u) = -\frac{1}{2}\sqrt{k/u^3}$$

So that the linear part of the replicating portfolio equals:

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$$V(m^0) + V'(m^0)(m^T - m^0) = 2R_\alpha^0 m^0 + R_\alpha^0(m^T - m^0) = R_\alpha^0(m^0 + m^T)$$

Interestingly this part is exactly $\overline{V}$. So the Impermanent Loss can be replicated by the Option part only:

$$L = -\frac{1}{2}\left(\int_{m^0}^{\infty} \sqrt{k/u^3}C(u)du + \int_0^{m^0} \sqrt{k/u^3}P(u)du\right)$$

L is always negative because of the concavity of the payoff function V. It means the LPs always lose money (albeit not a lot as the weights of the options $\sqrt{k/u^3}$ should be relatively small).

We can write L proportionally to the initial capital by noting that $\sqrt{k} = R_\alpha^0\sqrt{m^0}$:

$$L = (2R_\alpha^0 m^0)\left[-\frac{1}{4}\left(\int_{m^0}^{\infty} \frac{1}{\sqrt{m^0u^3}}C(u)du + \int_0^{m^0} \frac{1}{\sqrt{m^0u^3}}P(u)du\right)\right]$$

By a change of variable $v = u/m^0$ and denote the Call and Put payoff on relative performance as $C_p(v) = (r - v)^+, P_p(v) = (v - r)^+$ we get:

$$L = (2R_\alpha^0 m^0)\left[-\frac{1}{4}\left(\int_1^{\infty} \frac{1}{\sqrt{v^3}}C_p(v)dv + \int_0^1 \frac{1}{\sqrt{v^3}}P_v(v)dv\right)\right]$$

The Impermanent Loss represents a short volatility position from the LPs perspective. Its pricing formula looks strikingly similar to traditional instruments such as the Variance swaps or the CMS.